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ON THE SYMMETRIC FOURTH POWER L-FUNCTION OF GL_2

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ABSTRACT

In this paper I introduce a global Rankin–Selberg integral representing the symmetric fourth power L-function for GL(2). I show that the global integral is factorizable and compute the local unramified integrals. Finally, I also study some other properties of the local nonarchimedean integrals.

Introduction

Let π be a cusp form on $GL_2(\mathbb{A})$ and let χ be a unitary character of $F^*\backslash\mathbb{A}^*$. The Langlands program attaches to π the twisted L-function $L\left(\pi\otimes\chi, \mathrm{sym}^4, s\right)$ associated with the symmetric fourth power representation of $GL_2(\mathbb{C})$. This L-function is of degree five. My goal is to show that for generic χ the partial symmetric fourth power L-function is holomorphic and to use that for studying estimations of Hecke eigenvalues of Maass forms in the spirit of [1], [2] and [11]. This paper is a first step toward this goal. Here we first introduce, in section 2, a global integral which we show to be Eulerian. This Rankin–Selberg type integral involves a double cover Eisenstein series on the group G_2. It also involves a theta function, and hence might be viewed as a "Shimura type" integral. In the third section, we compute the unramified local integral obtained from the global one and show for nonarchimedean local fields that data could be chosen so that those integrals do not vanish. The next steps, which we hope to deal with in the near future, include the study of the archimedean local integrals and the global study

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of the poles of our Eisenstein series. Let us mention that this L-function was also studied in [11], but from a different point of view.

The integral discussed in this paper was announced in [5]. I wish to thank S. Rallis for helpful conversations.

1. Notations

1.1. Let G denote the exceptional group G_2. We denote its two simple roots by α, the short root and by β the long root. The positive roots of G are $\alpha, \beta, \alpha+\beta$, $2\alpha+\beta$, $3\alpha+\beta$, $3\alpha+2\beta$. If ε is a root $x_\varepsilon(r)$ denotes the one parameter unipotent subgroup corresponding to ε. The maximal split torus of G is denoted by $h(t_1, t_2)$ and parameterized such that

$$h^{-1}(t_1,t_2)x_\alpha(r)h(t_1,t_2) = x_\alpha(t_2^{-1}r),$$
$$h^{-1}(t_1,t_2)x_\beta(r)h(t_1,t_2) = x_\beta(t_1^{-1}t_2r).$$

Let W denote the Weyl group of G. The simple reflections w_α and w_β corresponding to the simple roots α and β satisfy the following:

$$w_\alpha h(t_1,t_2)w_\alpha^{-1} = h(t_1t_2, t_2^{-1}),$$
$$w_\beta h(t_1,t_2)w_\beta^{-1} = h(t_2, t_1).$$

Let $P = \mathrm{GL}_2 U$ (resp. $Q = \mathrm{GL}_2 V$) denote the maximal parabolic subgroup of G such that $x_\alpha(r) \subseteq \mathrm{GL}_2$ (resp. $x_\beta(r) \subseteq \mathrm{GL}_2$). U and V denote the corresponding unipotent radical subgroups of P and Q, respectively. In particular, $\dim U = \dim V = 5$. For more details and other group relations in G, see [4] and the references cited there.

We also recall the definition of τ in [4]. Let H_3 denote the Heisenberg group with three letters. Let $\overline{V}$ be the normal subgroup of V generated by $x_{3\alpha+\beta}(r_1)$ and $x_{3\alpha+2\beta}(r_2)$. It is not hard to check that $H_3 \simeq V/\overline{V}$. We define a homomorphism $\tau : V \to H_3$ to be the composite map of the projection from V to $V/\overline{V}$ with the above isomorphism.

As usual if H is an algebraic group and k a ring containing its field of definition, H_k or $H(k)$ will denote the k points of H.

1.2. In [8] Matumoto constructed a unique double cover for the group G_2 which we shall denote by $\widetilde{G}_2$ or $\widetilde{G}$. Denote by $(\,,)$ the two order Hilbert symbol. Let

$h_\alpha(a) = h(a^{-1}, a^2)$ and $h_\beta(b) = h(b, b^{-1})$. It follows from [8] that there is a cocycle σ on $G \times G$ such that

$$\sigma(h_\alpha(a)h_\beta(b), h_\alpha(c)h_\beta(d)) = (a,c)(b,d)(a,d).$$

We describe the restriction of σ to the maximal parabolic subgroups of G. For $g_1, g_2 \in \mathrm{GL}_2$ let $\kappa(g_1, g_2)$ denote the Kubota symbol (see [7]). Thus

$$\kappa(g_1,g_2) = \Big(\frac{\nu(g_1g_2)}{\nu(g_1)}, \frac{\nu(g_1g_2)}{\nu(g_2)}\Big)\Big(\det g_1, \frac{\nu(g_1g_2)}{\nu(g_1)}\Big),$$

where

$$\nu\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} c, & c \neq 0, \\ d, & c = 0. \end{cases}$$

It is not hard to check that the restriction of σ to the two Levi parts is the same and satisfies

$$\sigma(g_1, g_2) = \kappa(g_1,g_2)(\det g_1, \det g_2) \tag{1.2.1}$$

for all $g_1, g_2 \in \mathrm{GL}_2$ (in P or Q). In particular this means that the restriction of $\widetilde{G}$ to both Levi parts of P and Q yields a double cover of GL_2, denoted by $\widetilde{\mathrm{GL}_2}$. Following [8] we choose the covering so that σ is trivial on the maximal unipotent subgroup of $\widetilde{G}$ prescribed by the choice of the positive roots given in 1.1.

In general, if H is a reductive group, $\widetilde{H}$ will denote its two-fold cover. If $L \subset H$ is a subgroup, $\widetilde{L}$ will denote its full inverse image in $\widetilde{H}$. If there is a splitting homomorphism for L, we shall denote by L its image in $\widetilde{L}$ under this homomorphism. When needed we shall describe this homomorphism in detail. We shall also denote $\mathbf{s} : H \to \widetilde{H}$ the canonical section $\mathbf{s}(h) = \langle h, 1\rangle$ (here we identified $\widetilde{H}$ with the set of all pairs $\langle h, \varepsilon\rangle$ with $h \in H$ and $\varepsilon \in \{\pm 1\}$). When there is no confusion we shall write h for $\mathbf{s}(h)$.

1.3. In this section we recall some properties of the Weil representation. We refer the reader to [4] section 1.2 and the references there for complete details.

We identify elements $h \in H_3$ with triples (x,y,z). More precisely, let F be a global field and $\mathbf{A}$ its ring of adeles. Thus to each $h \in H_3(\mathbf{A})$ we attach a triple (x,y,z), where $x,y,z \in \mathbf{A}$. The product is given by

$$(x_1,y_1,z_1)(x_2,y_2,z_2) = (x_1+x_2, y_1+y_2, z_1+z_2+x_1y_2-y_1x_2).$$

Let ψ be a nontrivial additive character of $F\backslash\mathbb{A}$. Denote by $\mathcal{S}(\mathbb{A})$ the Schwartz functions on $\mathbb{A}$. The Weil representation ω_ψ is a representation of $H_3(\mathbb{A})\widetilde{\mathrm{SL}_2}(\mathbb{A})$ which acts on $\mathcal{S}(\mathbb{A})$. We have the following formulas:

$$\omega_\psi\left[(0,y,z)(x,0,0)\right]\phi(\xi)=\phi(\xi+x)\psi(\xi y+z), \tag{1.3.1}$$

$$\omega_\psi\left(\left\langle\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix},\varepsilon\right\rangle\right)\phi(\xi)=\varepsilon\frac{\gamma(1)}{\gamma(t)}|t|^{1/2}\phi(t\xi), \tag{1.3.2}$$

$$\omega_\psi\left(\left\langle\begin{pmatrix} 1 & x\\ & 1\end{pmatrix},\varepsilon\right\rangle\right)\phi(\xi)=\psi\left(\frac{1}{2}x\xi^2\right)\phi(\xi)\,. \tag{1.3.3}$$

Here $\phi\in\mathcal{S}(\mathbb{A})$, and $\gamma(t)$ for $t\in\mathbb{A}^*$, denotes the Weil constant and $\varepsilon\in\{\pm1\}$.

We define the theta function on $H_3(\mathbb{A})\widetilde{\mathrm{SL}_2}(\mathbb{A})$ by

$$\widetilde{\theta}_\phi(hg)=\sum_{\xi\in F}\omega_\psi(hg)\phi(\xi)$$

for all $h\in H_3(\mathbb{A})$, $g\in\widetilde{\mathrm{SL}_2}(\mathbb{A})$ and $\phi\in\mathcal{S}(\mathbb{A})$.

1.4. Let F and $\mathbb{A}$ be as 1.3. Let $\pi=\bigotimes_v\pi_v$ be an irreducible cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$. We shall denote by ω_π its central character. It is well known that π is a generic representation. More precisely, if we realize π in the space $L_0^2(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}))$, then the integral

$$W_\varphi(g)=\int\limits_{F\backslash\mathbb{A}}\varphi\left[\begin{pmatrix}1 & x\\ & 1\end{pmatrix}g\right]\psi\left(\frac{1}{2}x\right)dx,\quad \varphi\in V_\pi$$

is not identically zero for all φ. Here V_π denotes the realization space of π in $L_0^2(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}))$. Also $g\in\mathrm{GL}_2(\mathbb{A})$ and ψ is as defined in 1.3. We shall denote the space of all functions $W_\varphi(g)$ by $\mathcal{W}(\pi,\psi)$. Thus it is well known that $\mathcal{W}(\pi,\psi)$ factorizes into local components. In other words we have $\mathcal{W}(\pi,\psi)=\bigotimes_\nu\mathcal{W}(\pi_\nu,\psi_\nu)$. Here $\mathcal{W}(\pi_\nu,\psi_\nu)$ is the Whittaker model of π_ν corresponding to the character ψ_ν.

We shall also need to use the theta representation of $\widetilde{\mathrm{GL}_2}(\mathbb{A})$. We recall some details from [3]. Let $\chi=\bigotimes_\nu\chi_\nu$ be a character of $F^*\backslash A^*$. In [3], for each such χ, the theta representation θ_χ is constructed (r_χ is the notation of [3]). It follows from Proposition 8.1.1 in [3] that if χ is not totally even, then θ_χ is cuspidal. In other words, if $\chi_\nu(-1)=-1$ for at least one place then θ_χ is cuspidal. In any case it follows from [3] Proposition 8.2 that each θ_χ is distinguished, i.e. it has a unique Whittaker model. As for π, we shall denote by $\mathcal{W}(\theta_\chi,\psi)$ the Whitaker model for θ_χ. Thus we have $\mathcal{W}(\theta_\chi,\psi)=\bigotimes_\nu(\mathcal{W}(\theta_\chi^{(\nu)},\psi_\nu)$.

1.5. In this section we construct the Eisenstein series we use. Let θ_χ be the theta representation of $\widetilde{\mathrm{GL}_2}(\mathbb{A})$ as in 1.4. Let $\gamma(t)$ for $t \in \mathbb{A}^*$ denote the global Weil symbol. Thus $\gamma \circ \det$ is a character of $\widetilde{\mathrm{GL}_2}(\mathbb{A})$ which, when there is no confusion, we shall denote simply as γ. We view det as a function of $\widetilde{\mathrm{GL}_2}(\mathbb{A})$ by composing it with the projection $\widetilde{\mathrm{GL}_2}(\mathbb{A}) \longrightarrow \mathrm{GL}_2(\mathbb{A})$. We extend the representation $\theta_\chi \cdot (\gamma \circ \det)$ of $\widetilde{\mathrm{GL}_2}(\mathbb{A})$ to $\widetilde{P}(\mathbb{A})$ by letting it act trivially on $\widetilde{U}(\mathbb{A})$. Denote by δ_P the modulus function of $P(\mathbb{A})$. We view δ_P as a function of $\widetilde{P}(\mathbb{A})$ by composing it with the projection $\widetilde{P}(\mathbb{A}) \to P(\mathbb{A})$. Given $s \in \mathbb{C}$ we construct

$$I(\chi, s) = \mathrm{Ind}_{\widetilde{P}(\mathbb{A})}^{\widetilde{G}(\mathbb{A})} \theta_\chi(\gamma \circ \det)^{-1} \otimes \delta_P^s .$$

Thus $F_s^\chi \in I(\chi, s)$ is a smooth function $F_s^\chi : \widetilde{G}(\mathbb{A}) \longrightarrow V_{\theta_\chi}$ (the space of θ_χ) satisfying

$$F_s^\chi(pg) = \delta_P^s(p)\gamma(\det h)^{-1}\theta_\chi(h)F_s^\chi(g)$$

for all $p = hu \in \widetilde{P}(\mathbb{A})$ where $h \in \widetilde{\mathrm{GL}_2}(\mathbb{A})$ and $u \in \widetilde{U}(\mathbb{A})$, and all $g \in \widetilde{G}(\mathbb{A})$. To view the space as a scalar valued function, let $\ell : V_{\theta_\chi} \longrightarrow \mathbb{C}$ be a $\mathrm{GL}_2(F)$ invariant form and denote $\widetilde{f}_\chi(g, s) = \ell(F_s^\chi(g))$ for all $g \in \widetilde{G}(\mathbb{A})$ and $s \in \mathbb{C}$. Here we used the well known fact that there is a splitting homomorphism for $G(F)$ in $\widetilde{G}(\mathbb{A})$, and the fact that θ_χ is automorphic. We define the Eisenstein series on $\widetilde{G}(\mathbb{A})$ by

$$\widetilde{E}(g, \widetilde{f}_\chi, s) = \sum_{\gamma \in P(F)\backslash G(F)} \widetilde{f}_\chi(\gamma g, s) .$$

This series converges for $\mathrm{Re}(s)$ large and has a meromorphic continuation to the whole complex plane.

2. The global integral

We keep the notations of section 1. Let $\varphi \in V_\pi$. We view SL_2 as a subgroup of G_2 by embedding it in the Levi part of Q. We denote this embedding by j. We introduce our global integral,

$$(2.1)\quad I(\varphi, \phi, \widetilde{f}_\chi, s) = \int\limits_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} \int\limits_{V(F)\backslash V(\mathbb{A})} \varphi(g)\widetilde{\theta}_\phi(\tau(v)g)\widetilde{E}(vj(g), \widetilde{f}_\chi, s)dvdg .$$

Here $V(\mathbb{A})$ is embedded in $\widetilde{G}(\mathbb{A})$ as described in 1.2 and $j(g)$ stands for $\mathbf{s}(j(g))$ for $g \in \mathrm{SL}_2(\mathbb{A})$. The integral is well defined since $\widetilde{\theta}_\phi(\tau(v)g)\widetilde{E}(vj(g), \widetilde{f}_\chi, s)$ is not a genuine function of $\widetilde{\mathrm{SL}_2}(\mathbb{A})$. It also follows from the cuspidality of φ and from the fact that $\widetilde{\theta}_\phi(\tau(v)g)\widetilde{E}(vj(g),\widetilde{f}_\chi,s)$ is a slowly increasing function of g that $I(\varphi,\phi,\widetilde{f}_\chi,s)$ converges absolutely for all s for which the Eisenstein series has no poles.

Remark: We wish to mention that $I(\varphi,\phi,\widetilde{f}_\chi,s)$ is "dual" to the integral (2.1) in [4] in the sense that the cusp form and the Eisenstein series are interchanged.

For every $\widetilde{f}_\chi(g,s)$ as in subsection 1.5 we introduce the function

$$\widetilde{f}_{W_\chi}(h,s)=\int\limits_{F\backslash\mathbb{A}}\widetilde{f}_\chi(x_\alpha(r)h,s)\psi(r)dr \tag{2.2}$$

for all $h\in\widetilde{G}(\mathbb{A})$. Therefore $\widetilde{f}_{W_\chi}(h,s)$ belongs to the space $\mathrm{Ind}_{\widetilde{P}(\mathbb{A})}^{\widetilde{G}(\mathbb{A})}\left(\delta_P^s(\gamma\cdot\det)^{-1}\otimes\mathcal{W}(\theta_\chi,\psi)\right)$. In particular, due to the uniqueness of the Whittaker model of θ_χ the above induced representation is factorizable.

Finally, we shall denote $w_0=w_\beta w_\alpha w_\beta w_\alpha$ and we let N be the maximal unipotent subgroup of GL_2 consisting of upper triangular matrices. Then we have:

PROPOSITION 2.1: *For* $\mathrm{Re}(s)$ *large,*

$$\begin{aligned} I(\varphi,\phi,\widetilde{f}_\chi,s)=&\int\limits_{N(\mathbb{A})\backslash\mathrm{SL}_2(\mathbb{A})}\ \int\limits_{\mathbb{A}^4}W_\varphi(g)\omega_\psi(g)\phi(r_1+1)\\ &\widetilde{f}_{W_\chi}\big(w_0x_\alpha(r_1)x_{2\alpha+\beta}(r_2)x_{3\alpha+\beta}(r_3)x_{3\alpha+2\beta}(r_4)g,s\big)\psi(r_2)dr_idg\ .\end{aligned} \tag{2.3}$$

Proof: We shall carry out the process of unfolding formally. To justify the convergence of the integrals at each step when $\mathrm{Re}(s)$ is large, one can argue as in [6]. Thus assume $\mathrm{Re}(s)$ is large. Unfolding the Eisenstein series we see that $I(\varphi,\phi,f_\chi,s)$ equals

(2.4)

$$\sum_{\delta\in P(F)\backslash G(F)/\mathrm{SL}_2(F)V(F)}\ \int\limits_{(\mathrm{SL}_2(F)V(F))^\delta\backslash\mathrm{SL}_2(\mathbb{A})V(\mathbb{A})}\varphi(g)\widetilde{\theta}_\phi(\tau(v)g)\widetilde{f}_\chi(\delta vj(g),s)dvdg.$$

Here $(\mathrm{SL}_2V)^\delta=\delta^{-1}P\delta\cap\mathrm{SL}_2V$. It is not hard to check that $|P\backslash G/\mathrm{SL}_2V|=3$ and, as representatives, we may choose e, $w_\beta w_\alpha$ and w_0. We shall show that the first two representatives contribute zero to (2.4). Assume first that $\delta=e$. Thus $(\mathrm{SL}_2V)^\delta=P\cap\mathrm{SL}_2V\supset x_{2\alpha+\beta}(r)$ and $\widetilde{f}_\chi\left(\delta x_{2\alpha+\beta}(r)h,s\right)=\widetilde{f}_\chi(h,s)$ for all $r\in\mathbb{A}$ and $h\in\widetilde{G}(\mathbb{A})$. Hence we obtain in (2.4) as an inner integral

$$\int\limits_{F\backslash\mathbb{A}}\widetilde{\theta}_\phi\left((0,0;r)m\right)dr$$

where $(0,0,r)=\tau\left(x_{2\alpha+\beta}(r)\right)$ and $m\in H_3(\mathbf{A})\widetilde{\mathrm{SL}_2}(\mathbf{A})$. Using (1.3.1) this integral is zero. Next assume $\delta=w_\beta w_\alpha$. One can check that $(\mathrm{SL}_2(F)V(F))^\delta$ is generated by $h(t,t^{-1})$, $x_\beta(r_1), x_{\alpha+\beta}(r_2), x_{2\alpha+\beta}(r_3)$ and $x_{3\alpha+2\beta}(r_4)$ where $t\in F^*$ and $r_i\in F$. It is also not hard to check that

$$\delta x_\beta(r_1)\delta^{-1}=x_{3\alpha+2\beta}(r_1)\quad\text{and}\quad \delta x_{\alpha+\beta}(r_2)\delta^{-1}=x_{2\alpha+\beta}(r_2)\ .$$

Thus we get that

$$\widetilde{f}_\chi\left(\delta x_\beta(r_1)x_{\alpha+\beta}(r_2)h,s\right)=\widetilde{f}_\chi(h,s)$$

for all $r_1,r_2\in\mathbf{A}$ and $h\in\widetilde{G}_2(\mathbf{A})$. Hence we obtain in (2.4) as an inner integral

$$\text{(2.5)}\qquad \int\limits_{(F\backslash\mathbf{A})^2}\varphi\Big[\begin{pmatrix}1&r_1\\&1\end{pmatrix}g\Big]\widetilde{\theta}_\phi\Big[(0,r_2,0)\begin{pmatrix}1&r_1\\0&1\end{pmatrix}mg\Big]dr_1dr_2$$

where $(0,r_2,0)=\tau(x_{\alpha+\beta}(r_2))$, $m\in H_3(\mathbf{A})$ and $g\in\widetilde{\mathrm{SL}_2}(\mathbf{A})$. Using the definition of the theta function and (1.3.1) we get

$$\begin{aligned}\widetilde{\theta}_\phi\Big[(0,r_2,0)\begin{pmatrix}1&r_1\\&1\end{pmatrix}mg\Big]&=\sum_{\xi\in F}\omega_\psi\Big[(0,r_2,0)\begin{pmatrix}1&r_1\\&1\end{pmatrix}mg\Big]\phi(\xi)\\&=\sum_{\xi\in F}\psi(r_2\xi)\omega_\psi\Big[\begin{pmatrix}1&r_1\\&1\end{pmatrix}mg\Big]\phi(\xi)\ .\end{aligned}$$

Plugging this into (2.5) and integrating over r_2 first we see that (2.5) equals

$$\int\limits_{F\backslash\mathbf{A}}\varphi\Big[\begin{pmatrix}1&r_1\\&1\end{pmatrix}g\Big]\omega_\psi\Big[\begin{pmatrix}1&r_1\\&1\end{pmatrix}mg\Big]\phi(0)dr_1\ .$$

It follows from (1.3.3) that

$$\omega_\psi\Big[\begin{pmatrix}1&r_1\\&1\end{pmatrix}mg\Big]\phi(0)=\omega_\psi(mg)\phi(0),$$

and hence the above integral vanishes by cuspidality of φ. Thus we are left with $\delta=w_0$. We have

$$(\mathrm{SL}_2V)^\delta=h(t,t^{-1})x_\beta(r_1)x_{\alpha+\beta}(r_2)$$

and

$$\delta x_\beta(r_1)\delta^{-1}=x_{3\alpha+\beta}(r_1)\quad\text{and}\quad\delta x_{\alpha+\beta}(r_2)\delta^{-1}=x_\alpha(r_2)\ .$$

Thus

(2.6)

$$I(\varphi,\phi,\widetilde{f}_\chi,s)=\int\limits_{\mathrm{GL}_1(F)N(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})}\int\limits_{\mathbb{A}^4}\int\limits_{(F\backslash\mathbb{A})^2}\varphi\Big[\begin{pmatrix}1 & m_1\\ & 1\end{pmatrix}g\Big]\widetilde{\theta}_\phi$$
$$\Big[\tau(x_{\alpha+\beta}(m_2))\begin{pmatrix}1 & m_1\\ & 1\end{pmatrix}\tau\Big(x_\alpha(r_1)x_{2\alpha+\beta}(r_2)\Big)g\Big]$$
$$\widetilde{f}_\chi\Big[x_\alpha(m_2)w_0x_\alpha(r_1)x_{2\alpha+\beta}(r_2)x_{3\alpha+\beta}(r_3)x_{3\alpha+2\alpha}(r_4)j(g),s\Big]dm_1dm_2dr_idg.$$

It follows from the definition of the theta function that for any $h\in H_3(\mathbb{A})\widetilde{SL_2}(\mathbb{A})$

$$(2.7)\quad \widetilde{\theta}_\phi\Big[\tau(x_{\alpha+\beta}(m_2))\begin{pmatrix}1 & m_1\\ 0 & 1\end{pmatrix}h\Big]=\sum_{\xi\in F}\omega_\psi\Big[\tau(x_{\alpha+\beta}(m_2)\begin{pmatrix}1 & m_1\\ & 1\end{pmatrix}h\Big]\phi(\xi).$$

When $\xi=0$ the right hand side of (2.7) is invariant under $\begin{pmatrix}1 & m_1\\ & 1\end{pmatrix}$ and hence, by cuspidality of φ, the contribution to (2.6) for $\xi=0$ is zero. Thus in (2.7) we may sum over $\xi\in F^*$, i.e. $\xi\in\mathrm{GL}_1(F)$. Thus the only contribution from (2.7) to (2.6) is from

$$\sum_{\xi\in\mathrm{GL}_1(F)}\omega_\psi\Big[\begin{pmatrix}\xi^{-1} & \\ & \xi\end{pmatrix}\tau(x_{\alpha+\beta}(m_2))\begin{pmatrix}1 & m_1\\ & 1\end{pmatrix}h\Big]\phi(1)\ .$$

Plugging this into (2.6) and collapsing the summation with the integration we get

$$I(\varphi,\phi,\widetilde{f}_\chi,s)=\int\limits_{N(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})}\int\limits_{\mathbb{A}^4}\int\limits_{(F\backslash\mathbb{A})^2}\varphi\Big[\begin{pmatrix}1 & m_1\\ & 1\end{pmatrix}g\Big]$$
$$(2.8)\quad \omega_\psi\Big[\tau\Big(x_{\alpha+\beta}(m_2)\Big)\begin{pmatrix}1 & m_1\\ & 1\end{pmatrix}\tau\Big(x_\alpha(r_1)x_{2\alpha+\beta}(r_2)\Big)g\Big]\phi(1)$$
$$\widetilde{f}_\chi\Big[x_\alpha(m_2)w_0x_\alpha(r_1)x_{2\alpha+\beta}(r_2)x_{3\alpha+\beta}(r_3)x_{3\alpha+2\beta}(r_4)j(g),s\Big]dm_1dm_2dr_idg\ .$$

It follows from (1.3.1) and (1.3.3) that

$$\omega_\psi\Big[\tau\Big(x_{\alpha+\beta}(m_2)\Big)\begin{pmatrix}1 & m_1\\ & 1\end{pmatrix}h\Big]\phi(1)=\psi\Big(\frac{1}{2}m_1+m_2\Big)\omega_\psi(h)\phi(1)$$

for all $h\in H_3(\mathbb{A})\widetilde{\mathrm{SL}Sl_2}(\mathbb{A})$. Using this in (2.8), (2.3) follows. ∎

It follows from Proposition 2.1 that $I(\varphi, \phi, \widetilde{f}_\chi, s)$ is factorizable. More precisely, let

$$\pi = \bigotimes_\nu \pi_\nu,\ \omega_\psi = \bigotimes_\nu \omega_\psi^{(\nu)},\ \phi = \bigotimes_\nu \phi_\nu \text{ and } I(\chi, s) = \bigotimes_\nu I_\nu(\chi, s)$$

where ν runs over all places of F. Choose $\varphi \in V_\pi$ and $\widetilde{f}_\chi$ so that

$$W_\varphi = \bigotimes_\nu W_\nu \quad \text{and} \quad \widetilde{f}_{W_\chi} = \bigotimes_\nu \widetilde{f}^{(\nu)}_{W_\chi}$$

where $W_\nu \in \mathcal{W}(\pi_\nu, \psi_\nu)$ and

$$\widetilde{f}^{(\nu)}_{W_\chi} \in \operatorname{Ind}_{\widetilde{P}(F_\nu)}^{\widetilde{G}(F_\nu)} \left(\delta_P^s(\gamma \circ \det)^{-1} \bigotimes \mathcal{W}(\theta_\chi^{(\nu)}, \psi_\nu)\right).$$

Then

$$I(\varphi, \phi, \widetilde{f}_\chi, s) = \prod_\nu I_\nu(W_\nu, \phi_\nu, \widetilde{f}^{(\nu)}_{W_\chi}, s) \tag{2.9}$$

where

$$I_\nu(W_\nu, \phi_\nu, \widetilde{f}^{(\nu)}_{W_\chi}, s) = \int_{N(F_\nu)\backslash \mathrm{SL}_2(F_\nu)} \int_{F_\nu^4} W_\nu(g)\omega_\psi^{(\nu)}(g)\phi_\nu(r_1+1)$$
$$\widetilde{f}^{(\nu)}_{W_\chi}(w_0 x_\alpha(r_1)x_{2\alpha+\beta}(r_2)x_{3\alpha+\beta}(r_3)x_{3\alpha+2\beta}(r_4)j(g), s)\,\psi_\nu(r_2)dr_i dg\ .$$

The relation (2.9) holds provided the right hand side converges absolutely and provided that each local integral converges absolutely for Re(s) large. Indeed, in the next section we shall study these local integrals, computing the unramified integrals and proving other local properties we shall need.

3. The local theory

The subject of this section is to study the local integrals $I_\nu(W_\nu, \phi_\nu, f^{(\nu)}_{W_\chi}, s)$ at nonarchimedean places. We shall carry out the unramified computations and show the existence of data so that the local integrals do not vanish.

In this section $F = F_\nu$ will denote a nonarchimdean field. We shall drop the reference to ν from the notations when there is no confusion. Let π be an irreducible admissible generic representation of $\mathrm{GL}_2(F)$. Let ψ be a non-trivial additive character of F. We shall denote by $\mathcal{W}(\pi, \psi)$ the Whittaker model associated with π. We let ω_ψ denote the local oscillator representation on $H_3 \cdot \widetilde{\mathrm{SL}_2}$ and we denote by $\mathcal{S}(F)$ the space of Schwartz functions on F. The action of ω_ψ on $\mathcal{S}(F)$ is well known (see [9]). Let χ be a character of F^* and let θ_χ be the theta function as constructed in [3]. It follows from [3] that θ_χ is generic and we shall denote by $\mathcal{W}(\theta_\chi, \psi)$ its Whittaker model. Let $I(\mathcal{W}_{\theta_\chi}, s) = \mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}}\left(\delta_P^s(\gamma \cdot \det)^{-1} \bigotimes \mathcal{W}(\theta_\chi, \psi)\right)$. Thus a function $\widetilde{f}_{W_\chi}$ or $\widetilde{f}_\chi$ in short, in $I(\mathcal{W}_{\theta_\chi}, s)$ is a smooth function on the group $\widetilde{G}$ which takes values in $\mathcal{W}(\theta_\chi, \psi)$. More precisely, for any $h \in \widetilde{G}$ there is a function $W_{\chi,s}^h \in \mathcal{W}(\theta_\chi, \psi)$ such that for all $g \in \widetilde{GL_2} \subset \widetilde{P}$ and all $u \in \widetilde{U}$,

$$\widetilde{f}_\chi(guh, s) = W_{\chi,s}^h(g)\delta_P^s(g)\gamma^{-1}(\det g) .$$

In this section we will study the local integrals

$$I(W, \phi, \widetilde{f}_\chi, s) = \int_{N\backslash \mathrm{SL}_2} \int_{F^4} W(g)\omega_\psi(g)\phi(r_1 + 1)$$

$$\widetilde{f}_\chi(w_0 x_\alpha(r_1) x_{2\alpha+\beta}(r_2) x_{3\alpha+\beta}(r_3) x_{3\alpha+2\beta}(r_4) j(g), s)\psi(r_2) dr_i dg .$$

Let (,) denote the local quadratic Hilbert symbol. Let γ_t denote the local Weil factor. Thus $\gamma_a\gamma_b = (a,b)\gamma_{ab}$ for all $a, b \in F^*$. If F is nonarchimedean then $(\varepsilon, \mu) = 1$ if ε, μ are units and also $\gamma_\varepsilon = 1$. Some of the computations in this section will require some symbol computation. We remind the reader that $\sigma(g_1, g_2) = 1$ (see 1.2) if g_1 or g_2 are unipotent matrices of G corresponding to the positive roots which were chosen in section 1. Also, if g_1 and g_2 are in the Levi part of P or Q we may use, for computing $\sigma(g_1, g_2)$, formula (1.2.1).

Finally, we will denote by $\mathcal{O}$ the ring of integers and by $\mathcal{O}^*$ the units in $\mathcal{O}$. Also, p will denote a generator of the maximal ideal in $\mathcal{O}$ and $|p| = q^{-1}$. For any local field, $K(H)$ will denote the standard maximal compact subgroup of H where H is a reductive group. There is a splitting homomorphism for $K(G)$ in $\widetilde{G}$ and if there is no confusion, we shall identify $K(G)$ and its subgroups with its image in $\widetilde{G}$.

3.1. THE UNRAMIFIED COMPUTATION. In this section we shall carry out the unramified computation. We assume that q is odd. We assume that there exists a vector $W \in \mathcal{W}(\pi, \psi)$ such that $W(k) = W(e) = 1$ for all $k \in K(\mathrm{GL}_2)$. Such a vector W is unique. In this case ψ is an additive character of F which is trivial on $\mathcal{O}$. In a similar way, we assume that $\mathcal{W}(\theta_\chi, \psi)$ contains a unique vector W_χ such that $W_\chi(k) = W_\chi(e) = 1$ where $k \in K(\mathrm{GL}(2))$ viewed as a subgroup of $\widetilde{\mathrm{GL}(2)}$ in the usual way [7]. More precisely the splitting homomorphism is described by $k \to \langle k, \wedge(k)\rangle$ where

$$\wedge\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} (c, d(ad-bc)), & 0 < |c| < 1 \\ 1, & |c| = 0, 1 \end{cases} \tag{3.1}$$

for all $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K(\mathrm{GL}(2))$. Thus χ is unramified. To complete the choice of data we let $\widetilde{f}_\chi \in I(\mathcal{W}_{\theta_\chi}, s)$ be the unramified vector in this space and ϕ the Schwartz function on F which equals one on $\mathcal{O}$ and zero otherwise. Thus ϕ is fixed under $K(\mathrm{SL}_2)$ viewed as a subgroup in $\widetilde{\mathrm{SL}_2}$.

Next we describe the local L-function we study. By our assumption on π we may assume that $\pi = \mathrm{Ind}_{B_2}^{\mathrm{GL}_2}(\mu_1, \mu_2)$ where μ_1, μ_2 are unramified. Here B_2 is the Borel subgroup of GL_2 which consists of upper triangular matrices and

$$(\mu_1, \mu_2)\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \mu_1(a)\mu_2(c)\left|\frac{a}{c}\right|^{1/2}.$$

From general theory we may associate to π a semisimple conjugacy class in $\mathrm{GL}_2(\mathbb{C})$ which we can choose to be $\mathrm{diag}(\mu_1(p), \mu_2(p))$. Let sym^4 denote the symmetric fourth power representation of $\mathrm{GL}_2(\mathbb{C})$. It is well known that sym^4 is an orthogonal representation. Denote

$$\begin{aligned} A(p) &= \mathrm{sym}^4\begin{pmatrix} \mu_1(p) & \\ & \mu_2(p) \end{pmatrix} \\ &= \mathrm{diag}\left(\mu_1^2\mu_2^{-2}(p), \mu_1\mu_2^{-1}(p), 1, \mu_1^{-1}\mu_2(p), \mu_1^{-2}\mu_2^2(p)\right). \end{aligned}$$

We define the local twisted symmetric fourth power L-function to be

$$L(\pi \otimes \chi, \mathrm{sym}^4, s) = \det\left[I_5 - \chi(p)A(p)q^{-s}\right]^{-1}.$$

Here I_5 is the 5×5 identity matrix. Finally we denote by

$$L(\chi, s) = (1 - \chi(p)q^{-s})^{-1}$$

the local Dirichlet L-function associated with χ.

In this section we prove

PROPOSITION 3.1: *For all unramified data as above and for* $\mathrm{Re}(s)$ *large enough,*

$$I(W,\phi,\widetilde{f}_\chi,s)=\frac{L(\pi\otimes\chi,\mathrm{sym}^4,6s-\frac{5}{2})}{L(\chi,6s-\frac{3}{2})L(\chi^2,12s-5)L(\chi^3,18s-\frac{15}{2})}. \tag{3.2}$$

Proof: We start by computing the integral $I(W,\phi,\widetilde{f}_\chi,s)$. We normalize the additive measure so that $\int_{\mathcal{O}}dx=1$. Using the Iwasawa decomposition for SL_2 we get

$$I(W,\phi,\widetilde{f}_\chi,s)=\int\limits_{F^*}\int\limits_{F^4}W\begin{pmatrix}t&\\&t^{-1}\end{pmatrix}\omega_\psi\begin{pmatrix}t&\\&t^{-1}\end{pmatrix}\phi(r_1+1)$$
$$\widetilde{f}_\chi\left[w_0x_\alpha(r_1)x_{2\alpha+\beta}(r_2)x_{3\alpha+\beta}(r_3)x_{3\alpha+2\beta}(r_4)h(t,t^{-1}),s\right]|t|^{-2}dr_id^*t\ .$$

Here we chose the measure $\int_{\mathcal{O}^*}d^*t=1$ and we used the $K(\mathrm{SL}_2)$-invariant property of the functions with the choice of measure $\int_{K(\mathrm{SL}_2)}dk=1$. Conjugating the torus, in the above identity, to the left we obtain

$$I(W,\phi,\widetilde{f}_\chi,s)=\int\limits_{F^*}\int\limits_{F^4}W\begin{pmatrix}t&\\&t^{-1}\end{pmatrix}\phi(r_1+t)(t,t)|t|^{-5/2}\gamma_t$$
$$\widetilde{f}_\chi\left[h(1,t)w_\beta w_\alpha w_\beta w_\alpha x_\alpha(r_1)x_{2\alpha+\beta}(r_2)x_{3\alpha+\beta}(r_3)x_{3\alpha+2\beta}(r_4),s\right]\psi(r_2)dr_id^*t\ .$$

We explain the above identity in more detail. First we obtain a factor of $|t|^{-1}$ from a change of variables $r_1\to t^{-1}r_1$, $r_3\to t^{-1}r_3$ and $r_4\to tr_4$ which results from the torus conjugation. The identity $h(1,t)w_\beta w_\alpha w_\beta w_\alpha=w_\beta w_\alpha w_\beta w_\alpha h(t,t^{-1})$ contributes a cocycle (t,t) which is obtained from the GL_2 identity

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}a&\\&b\end{pmatrix}=\begin{pmatrix}b&\\&a\end{pmatrix}\begin{pmatrix}&1\\1&\end{pmatrix}$$

using (1.2.1). Finally, we get the contribution of $|t|^{1/2}\gamma_t$ from the local version of (1.3.2). Since

$$W\begin{pmatrix}t&\\&t^{-1}\end{pmatrix}=0\quad\text{for } |t|>1\quad\text{and}\quad\phi(r)=0\quad\text{for } |r|>1,$$

it follows that $I(w,\phi,f_\chi,s)$ vanishes on the domain $|r_1|>1$. Conjugating $w_\alpha x_\alpha(r_1)$ to the right we obtain

$$I(W,\phi,\widetilde{f}_\chi,s)=\int\limits_{|t|\le 1}\int\limits_{F^3}W\begin{pmatrix}t&\\&t^{-1}\end{pmatrix}(t,t)|t|^{-5/2}\gamma_t$$
$$\widetilde{f}_\chi\left[h(1,t)w_\beta w_\alpha w_\beta x_{\alpha+\beta}(r_2)x_\beta(r_3)x_{3\alpha+2\beta}(r_4),s\right]\psi(r_2)dr_id^*t\ .$$

Write

$$\int_{|t|\leq 1} = \sum_{n=0}^{\infty} \int_{|t|=q^{-n}} .$$

For $|\eta| = 1$, the identity $h(1,p^n\eta) = h(1,p^n)h(1,\eta)$ contributes a symbol (p^n,η). Using the identities $\gamma_{p^n\eta} = (p^n,\eta)\gamma_{p^n}$ and $(p^n\eta,p^n\eta) = (p^n,p^n)$ we obtain

$$(3.3)\qquad I(W,\phi,\widetilde{f}_\chi,s) = \sum_{n=0}^{\infty} W\begin{pmatrix} p^n & \\ & p^{-n}\end{pmatrix}(p^n,p^n)q^{\frac{5}{2}n}\gamma_{p^n}$$
$$\int_{F^3} \widetilde{f}_\chi\left[h(1,p^n)w_\beta w_\alpha w_\beta x_{\alpha+\beta}(r_2)x_\beta(r_3)x_{3\alpha+2\beta}(r_4),s\right]\psi(r_2)dr_i .$$

Set

$$J(n) = \int_{F^3} \widetilde{f}_\chi\left[h(1,p^n)w_\beta w_\alpha w_\beta x_{\alpha+\beta}(r_1)x_\beta(r_2)x_{3\alpha+2\beta}(r_3),s\right]\psi(r_1)dr_i.$$

In the following series of lemmas we will compute $J(n)$ and the right hand side of (3.3). Let

$$J_1(n) = \int_{F^2} \widetilde{f}_\chi\Big[h(1,p^n)w_\beta w_\alpha w_\beta x_\beta(r_2)x_{3\alpha+2\beta}(r_3),s\Big]dr_i$$

and

$$J_2(n) = \int_{F^2}\int_{|r_1|>1} \widetilde{f}_\chi\left[h(1,p^n)w_\beta w_\alpha w_\beta x_{\alpha+\beta}(r_1)x_\beta(r_2)x_{3\alpha+2\beta}(r_3),s\right]\psi(r_1)dr_i .$$

Then dividing the domain of integration of $J(n)$ into $|r_1| \leq 1$ and $|r_1| > 1$ yields the expression $J(n) = J_1(n) + J_2(n)$.

LEMMA 3.2: $J_1(2n+1) = 0$.

Proof: Using the right invariance of $\widetilde{f}_\chi$ by $h(\varepsilon^{-1},\varepsilon)$ with $\varepsilon \in \mathcal{O}^*$ we obtain

$$J_1(n) = \int_{F^2}\int_{|\varepsilon|=1} \widetilde{f}_\chi\left[h(1,p^n)w_\beta w_\alpha w_\beta x_\beta(r_2)x_{3\alpha+2\beta}(r_3)h(\varepsilon^{-1},\varepsilon),s\right]d^*\varepsilon dr_i.$$

Here we chose the measure $d^*\varepsilon$ so that $\int_{\mathcal{O}^*} d^*\varepsilon = 1$. Conjugating $h(\varepsilon^{-1},\varepsilon)$ to the left, we obtain

$$J_1(n) = \int_{F^2}\int_{|\varepsilon|=1} \widetilde{f}_\chi\Big[h(1,p^n)h(\varepsilon,1)w_\beta w_\alpha w_\beta x_\beta(r_2)x_{3\alpha+2\beta}(r_3),s\Big]d^*\varepsilon dr_i .$$

Before we proceed we recall the following formulas for the function W_χ (see [3]). We have

$$(3.4)\qquad \begin{aligned} W_\chi\begin{pmatrix} a & \\ & a\end{pmatrix} &= \chi(a)\gamma_a, \quad a\in F^*;\\ W_\chi\begin{pmatrix} p^n\varepsilon & \\ & 1\end{pmatrix} &= \begin{cases}\chi(p)^{\frac{n}{2}}q^{-\frac{n}{4}}, & n\equiv 0(2),\\ 0, & n\equiv 1(2),\end{cases}\end{aligned}$$

where $n\geq 0$ and $\varepsilon\in\mathcal{O}^*$.

Going back to $J_1(n)$, the identity $h(1,p^n)h(\varepsilon,1)=h(\varepsilon,1)h(1,p^n)$ contributes the symbol (p^n,ε) as can be seen from the GL_2 calculation. Since $h(\varepsilon,1)$ is in the center of GL_2 which is the Levi part of P, we can use the first formula in (3.4) to get

$$J_1(n)=\int\limits_{F^2}\widetilde{f}_\chi\left[h(1,p^n)w_\beta w_\alpha w_\beta x_\beta(r_2)x_{3\alpha+2\beta}(r_3),s\right]dr_i\int\limits_{|\varepsilon|=1}(p^n,\varepsilon)d^*\varepsilon\ .$$

Lemma 3.2 then follows since the last integral vanishes unless n is even.

Next we study $J_2(n)$. We start with:

Lemma 3.3: *Let* $G(p)=\sum\limits_{\varepsilon\in(\mathcal{O}/\mathcal{P})^*}(p,\varepsilon)\psi(p^{-1}\varepsilon)$ *and set*

$$R(n)=\int\limits_{F^2}\widetilde{f}_\chi\left[h(1,p^n)w_\beta w_\alpha w_\beta x_{\alpha+\beta}(p^{-1})x_\beta(r_2)x_{3\alpha+2\beta}(r_3),s\right]dr_i\ .$$

Then $J_2(2n)=-R(2n)$ *and* $J_2(2n+1)=G(p)R(2n+1)$.

Proof: We have,

$$J_2(n)=\int\limits_{F^2}\sum_{m=1}^{\infty}q^m$$
$$\int\limits_{|\varepsilon|=1}\widetilde{f}_\chi\Big[h(1,p^n)w_\beta w_\alpha w_\beta x_{\alpha+\beta}(p^{-m}\varepsilon)x_\beta(r_2)x_{3\alpha+2\beta}(r_3),s\Big]\psi(p^{-m}\varepsilon)d\varepsilon dr_i.$$

It follows from section 1.1 that

$$h(\varepsilon,\varepsilon^{-1})x_{\alpha+\beta}(p^{-m})h(\varepsilon^{-1},\varepsilon)=x_{\alpha+\beta}(p^{-m}\varepsilon)\ .$$

Thus,

$$J_2(n)=\int\limits_{F^2}\sum_{m=1}^{\infty}q^m$$
$$\int\limits_{|\varepsilon|=1}\widetilde{f}_\chi\Big[h(1,p^n)h(\varepsilon,1)w_\beta w_\alpha w_\beta x_{\alpha+\beta}(p^{-m})x_\beta(r_2)x_{3\alpha+2\beta}(r_3),s\Big]\psi(p^{-m}\varepsilon)d\varepsilon dr_i.$$

As in Lemma 3.2 we may conjugate $h(\varepsilon, 1)$ to the left, and use (3.4) to obtain

$$J_2(n) = \int_{F^2} \sum_{m=1}^{\infty} q^m \widetilde{f}_\chi \Big[h(1,p^n) w_\beta w_\alpha w_\beta x_{\alpha+\beta}(p^{-m}) x_\beta(r_2) x_{3\alpha+2\beta}(r_3), s\Big] dr_i \int_{|\varepsilon|=1} (p^n, \varepsilon)\psi(p^{-m}\varepsilon) d\varepsilon.$$

It is not hard to check that $\int_{|\varepsilon|=1}(p^n,\varepsilon)\psi(p^{-m}\varepsilon)d\varepsilon$ vanishes if $m > 1$. Indeed this follows from the fact that elements in $1+\mathcal{P}$ are squares and that the Hilbert symbol is trivial on squares. For $m = 1$, we have

$$\int_{|\varepsilon|=1} (p^n,\varepsilon)\psi(p^{-1}\varepsilon)d\varepsilon = \begin{cases} -q^{-1}, & n \equiv 0(2), \\ q^{-1}G(p), & n \equiv 1(2). \end{cases}$$

This proves the lemma.

To summarize, we have shown that

(3.5)

$$I(W,\phi,\widetilde{f}_\chi,s) = \sum_{n=0}^{\infty} W\begin{pmatrix} p^{2n} & \\ & p^{-2n}\end{pmatrix} q^{5n}(J_1(2n) - R(2n)) + (p,p)\gamma_p G(p) \sum_{n=0}^{\infty} W\begin{pmatrix} p^{2n+1} & \\ & p^{-(2n+1)}\end{pmatrix} q^{\frac{5}{2}(2n+1)} R(2n+1)\ .$$

We will show that the right hand side of (3.5) equals the right hand side of (3.2).

LEMMA 3.4:

$$J_1(2n) - R(2n) = \begin{cases} 1-\chi(p)q^{-6s+3/2} & n=0 \\ \frac{1-\chi(p)q^{-6s+3/2}}{1-\chi(p)q^{-6s+5/2}}\chi^n(p)q^{-(6s+1/2)n}\Big(1-\chi^{n+1}(p)q^{(-6s+5/2)(n+1)}\Big) & n\geq 1 \end{cases}$$

and

$$R(2n+1) = \begin{cases} 0 & n=0 \\ \frac{1-\chi(p)q^{-6s+3/2}}{1-\chi(p)q^{-6s+5/2}}(p,p)q^{-12s+3-(6s+1/2)n}\chi^{n+2}(p)\Big(1-\chi^n(p)q^{(-6s+5/2)n}\Big) & n\geq 1. \end{cases}$$

Let us show first how this lemma implies Proposition 3.1. Indeed let

$$K\begin{pmatrix} a & \\ & b\end{pmatrix} = |\tfrac{a}{b}|^{-1/2} W\begin{pmatrix} a & \\ & b\end{pmatrix}.$$

Using this and Lemma 3.4, equality (3.5) reads

$$I(W,\phi,\widetilde{f}_\chi,s)=\left(1-\chi(p)q^{-6s+3/2}\right)+\frac{1-\chi(p)q^{-6s+3/2}}{1-\chi(p)q^{-6s+5/2}}$$
$$\Big[\sum_{n=1}^{\infty}K\begin{pmatrix}p^{2n}&\\&p^{-2n}\end{pmatrix}q^{(-6s+5/2)n}\chi^n(p)\left(1-\chi^{n+1}(p)q^{(-6s+5/2)(n+1)}\right)$$
$$+\sum_{n=1}^{\infty}K\begin{pmatrix}p^{2n+1}&\\&p^{-(2n+1)}\end{pmatrix}q^{(-6s+5/2)n-12s+5}\chi^{n+2}(p)\left(1-\chi^n(p)q^{(-6s+5/2)n}\right)\Big]$$

where here we used the identity $\gamma_p G(p)=q^{1/2}$ (see [10]). Set $y=\chi(p)q^{-6s+5/2}$. Thus

$$I(W,\phi,\widetilde{f}_\chi,s)=\frac{1-yq^{-1}}{1-y}\Big[\sum_{n=0}^{\infty}K\begin{pmatrix}p^{2n}&\\&p^{-2n}\end{pmatrix}y^n(1-y^{n+1})$$
$$+\sum_{n=0}^{\infty}K\begin{pmatrix}p^{2n+1}&\\&p^{-(2n+1)}\end{pmatrix}y^{n+2}(1-y^n)\Big]\,.$$

Recall the Shintani [12] formula

$$K\begin{pmatrix}a&\\&a^{-1}\end{pmatrix}=\frac{\mu_1\mu_2^{-1}(a)-\mu_1^{-1}\mu_2(p)\mu_1^{-1}\mu_2(a)}{1-\mu_1^{-1}\mu_2(p)}\,.$$

Let $\alpha=\mu_1^{-1}\mu_2(p)$. Then

$$K\begin{pmatrix}p^m&\\&p^{-m}\end{pmatrix}=\frac{\alpha^{-m}-\alpha^{m+1}}{1-\alpha}.$$

Plugging this in the above equality we get

$$I(W,\phi,\widetilde{f}_\chi,s)=\frac{1-yq^{-1}}{(1-y)(1-\alpha)}\Big[\sum_{n=0}^{\infty}(\alpha^{-2n}-\alpha^{2n+1})(y^n-y^{2n+1})$$
$$+\sum_{n=0}^{\infty}\left(\alpha^{-(2n+1)}-\alpha^{2n+2}\right)\left(y^{n+2}-y^{2n+2}\right)\Big]\,.$$

Opening parentheses and summing the geometric series we obtain

$$I(W,\phi,\widetilde{f}_\chi,s)=\frac{(1-yq^{-1})(1-y^2)(1-y^3)}{(1-\alpha^2y)(1-\alpha y)(1-y)(1-\alpha^{-1}y)(1-\alpha^{-2}y)},$$

and this completes the proof of Proposition 3.1. ∎

To prove Lemma 3.4 we start with

LEMMA 3.5: *Set*

$$I(n) = \int_F \widetilde{f}_\chi \left[h(1,p^n) w_\beta \chi_\beta(r), s\right] dr \, .$$

Then,

$$I(2n) = q^{-(6s+1/2)n} \chi^n(p) \left[1 + (1-q^{-1})\chi(p) q^{-6s+5/2} \frac{1 - \chi^n(p) q^{(-6s+5/2)n}}{1 - \chi(p) q^{-6s+5/2}}\right]$$

and $I(2n+1) = 0$.

Proof: For all $r \neq 0$ we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & r^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r^{-1} & \\ & r \end{pmatrix} \begin{pmatrix} -1 & 0 \\ r^{-1} & 1 \end{pmatrix} .$$

If $|r| > 1$ then $\begin{pmatrix} -1 & 0 \\ r^{-1} & 1 \end{pmatrix}$ is in the maximal compact of GL_2. We also obtain a cocycle contribution of (r,r) due to the above factorization in $\widetilde{GL_2}$. We separate the domain of integration in $I(n)$ into $|r| \leq 1$ and $|r| > 1$. Since

$$\bigwedge \begin{pmatrix} -1 & 0 \\ r^{-1} & 1 \end{pmatrix} = (r,r) \quad \text{if } |r| > 1$$

we obtain

$$I(n) = \widetilde{f}_\chi\left(h(1,p^n)\right) + \int_{|r|>1} \widetilde{f}_\chi \left[h(1,p^n) \chi_\beta(r^{-1}) h(r^{-1}, r), s\right] dr \, .$$

It is not hard to check that

$$\delta_P\left(h(t_1,t_2)\right) = |t_1^6 t_2^3|. \tag{3.6}$$

Using this and the left invariance of $\widetilde{f}_\chi$ under x_β we get

$$I(n) = \gamma_{p^n}^{-1} q^{-3ns} W_\chi \begin{pmatrix} p^n & \\ & 1 \end{pmatrix} + \int_{|r|>1} \widetilde{f}_\chi \left[h(r^{-1}, p^n r), s\right] dr$$

or

$$I(n) = \gamma_{p^n}^{-1} q^{-3ns} W_\chi \begin{pmatrix} p^n & \\ & 1 \end{pmatrix} + \int_{|r|>1} W_\chi \begin{pmatrix} p^n & \\ & r^{-1} \end{pmatrix} \gamma_{p^n r^{-1}}^{-1} |r|^{-3s} q^{-3ns} dr \, .$$

Using the relation $\gamma_{p^n r^{-1}} = \gamma_{p^n}\gamma_{r^{-1}}(p^n, r^{-1})$ and

$$W_\chi \begin{pmatrix} p^n & \\ & r^{-1} \end{pmatrix} = \chi^{-1}(r)\gamma_{r^{-1}} W_\chi \begin{pmatrix} rp^n & \\ & 1 \end{pmatrix}$$

(see (3.4)) we obtain

$$I(n) = \gamma_{p^n}^{-1} q^{-3ns} W_\chi \begin{pmatrix} p^n & \\ & 1 \end{pmatrix} + \gamma_{p^n}^{-1} q^{-3ns} \int\limits_{|r|>1} W_\chi \begin{pmatrix} p^n r & \\ & 1 \end{pmatrix} \chi^{-1}(r)(p^n, r)|r|^{3s+1} d^* r.$$

Notice the change to the multiplicative measure. Thus

$$\begin{aligned} I(n) = \gamma_{p^n}^{-1} q^{-3ns} \Big[& W_\chi \begin{pmatrix} p^n & \\ & 1 \end{pmatrix} \\ & + \sum_{k=1}^{\infty} \chi^k(p) q^{(-3s+1)k} \int\limits_{|\varepsilon|=1} W_\chi \begin{pmatrix} p^{n-k}\varepsilon & \\ & 1 \end{pmatrix} (p^n, p^{-k}\varepsilon) d\varepsilon \Big] \\ = \gamma_{p^n}^{-1} q^{-3ns} \Big[& W_\chi \begin{pmatrix} p^n & \\ & 1 \end{pmatrix} \\ & + \sum_{k=1}^{\infty} \chi^k(p) q^{(-3s+1)k} W_\chi \begin{pmatrix} p^{n-k} & \\ & 1 \end{pmatrix} (p^n, p^k) \int\limits_{|\varepsilon|=1} (p^k, \varepsilon) d\varepsilon \Big]. \end{aligned}$$

Since $\int_{|\varepsilon|=1} (p^k, \varepsilon) d\varepsilon = 0$ when $k \equiv 1(2)$, we get

$$I(n) = \gamma_{p^n}^{-1} q^{-3ns} \Big[W_\chi \begin{pmatrix} p^n & \\ & 1 \end{pmatrix} + (1 - q^{-1}) \sum_{k=1}^{\infty} \chi(p)^{2k} q^{(-6s+2)k} W_\chi \begin{pmatrix} p^{n-2k} & \\ & 1 \end{pmatrix} \Big].$$

It follows from (3.4) that

$$W_\chi \begin{pmatrix} p^n & \\ & 1 \end{pmatrix} = W_\chi \begin{pmatrix} p^{n-2k} & \\ & 1 \end{pmatrix} = 0 \quad \text{if } n \equiv 1(2).$$

Thus $I(2n+1) = 0$. Also,

$$I(2n) = q^{-6ns} \Big[W_\chi \begin{pmatrix} p^{2n} & \\ & 1 \end{pmatrix} + (1 - q^{-1}) \sum_{k=1}^{\infty} \chi(p)^{2k} q^{(-6s+2)k} W_\chi \begin{pmatrix} p^{2(n-k)} & \\ & 1 \end{pmatrix} \Big].$$

If $n < k$ then $W_\chi\begin{pmatrix} p^{2(n-k)} & \\ & 1\end{pmatrix} = 0$ and, using (3.4), we obtain

$$\begin{aligned} I(2n) =& q^{-6ns}\left[q^{-\frac{n}{2}}\chi(p)^n + (1-q^{-1})\sum_{k=1}^{n} q^{-\frac{n-k}{2}+(-6s+2)k}\chi(p)^{n+k}\right] \\ =& q^{-(6s+1/2)n}\chi(p)^n\left[1+(1-q^{-1})\sum_{k=1}^{n} q^{(6s+s/2)k}\chi(p)^k\right] . \end{aligned}$$

Using the formula for geometric sums Lemma 3.5 follows. ∎

LEMMA 3.6: *We have*

$$J_1(2n) = I(2n)\frac{1-\chi^2(p)q^{-12s+3}}{1-\chi^2(p)q^{-12s+4}}.$$

Proof: By breaking up the domain of integration in r_1, we obtain

$$\begin{aligned} J_1(2n) =& \int_{F^2} \widetilde{f}_\chi\left[h(1,p^{2n})w_\beta w_\alpha w_\beta x_\beta(r_1)x_{3\alpha+2\beta}(r_2), s\right]dr_1 dr_2 \\ =& \int_F \widetilde{f}_\chi\left[h(1,p^{2n})w_\beta x_\beta(r_2), s\right]dr_2 \\ &+ \int_F \int_{|r_1|>1} \widetilde{f}_\chi\left[h(1,p^{2n})w_\beta x_\beta(r_2) w_\alpha x_\beta(r_1^{-1})h(r_1^{-1},r_1), s\right]dr_1 dr_2 \\ =& I(2n) + \int_F \int_{|r_1|>1} \widetilde{f}_\chi\left[h(1,p^{2n})w_\beta x_\beta(r_2)h(1,r_1^{-1}), s\right]dr_1 dr_2 . \end{aligned}$$

Here we used the Iwasawa decomposition for $w_\beta x_\beta(r_1)$ when $|r_1| > 1$ (which corresponds to the usual decomposition in GL_2; see the beginning of the proof of Lemma 3.5). We also conjugated $x_\beta(r_1^{-1})$ to the left and used the right and left invariance properties of $\widetilde{f}_\chi$. Thus, conjugating $h(1,r_1^{-1})$ to the left we get

$$\begin{aligned} J_1(2n) = I(2n) +& \int_F \int_{|r_1|>1} \widetilde{f}_\chi\left[h(1,p^{2n})h(r_1^{-1},1)w_\beta x_\beta(r_2 r_1^{-1}), s\right]dr_1 dr_2 \\ = I(2n) +& \left(\int_F f\left[h(1,p^{2n})w_\beta x_\beta(r_2), s\right]dr_2\right) \\ &\times \left(\int_{|r_1|>1} |r_1|^{-6s+1}\chi(r_1)^{-1}\gamma_{r_1^{-1}}dr_1\right) . \end{aligned}$$

This follows from the fact that $h(r_1^{-1},1)$ is in the center of the GL_2 which is the Levi part of P (see the proof of Lemma 3.2). Hence

$$\begin{aligned} J_1(2n) &= I(2n)\Big(1+\int\limits_{|r_1|>1}|r_1|^{-6s+2}\chi(r_1)^{-1}\gamma_{r_1^{-1}}dr_1\Big)\\ &= I(2n)\Big(1+\sum_{k=1}^{\infty}q^{(-6s+2)k}\chi(p)^k\gamma_{p^k}\int\limits_{|\varepsilon|=1}(p^k,\varepsilon)d\varepsilon\Big)\ .\end{aligned}$$

Thus we may sum over $k\equiv 0(2)$. We get

$$J_1(2n)=I(2n)\Big[1+(1-q^{-1})\sum_{k=1}^{\infty}q^{(-12s+4)k}\chi(p)^{2k}\Big]$$

and the lemma follows. ∎

To complete the proof of Lemma 3.4 we need to compute $R(m)$. Write $R(m)=R_1(m)+R_2(m)$ where

$$R_1(m)=\int\limits_F\int\limits_{|r_1|\le 1}\widetilde{f}_\chi\Big[h(1,p^m)w_\beta w_\alpha w_\beta x_{\alpha+\beta}(p^{-1})x_\beta(r_1)x_{3\alpha+2\beta}(r_2),s\Big]dr_1dr_2$$

and

$$R_2(m)=\int\limits_F\int\limits_{|r_1|>1}\widetilde{f}_\chi\Big[h(1,p^m)w_\beta w_\alpha w_\beta x_{\alpha+\beta}(p^{-1})x_\beta(r_1)x_{3\alpha+2\beta}(r_2),s\Big]dr_1dr_2\ .$$

We start with $R_1(m)$:

$$\begin{aligned} R_1(m) &= \int\limits_F\widetilde{f}_\chi\Big[h(1,p^m)w_\beta x_\beta(r_2)w_\alpha x_\alpha(p^{-1}),s\Big]dr_2\\ &= \int\limits_F\widetilde{f}_\chi\Big[h(1,p^m)w_\beta x_\beta(r_2)x_\alpha(p)h(p^{-1},p^2),s\Big]dr_2\\ &= \int\limits_F\widetilde{f}_\chi\Big[h(1,p^m)w_\beta x_\beta(r_2)h(p^{-1},p^2),s\Big]\psi\left(p^{m+1}r_2\right)dr_2\ .\end{aligned}$$

In the first equality we used the corresponding Iwasawa decomposition for the matrix $w_\alpha x_\alpha(p^{-1})$ (as in Lemma 3.5). The second equality follows from the relation

$$x_\beta(r_2)x_\alpha(p)=x_{\alpha+\beta}(pr_2)ux_\alpha(p)x_\beta(r_2)$$

where u is a unipotent matrix such that

$$\widetilde{f}_\chi\Big[h(1,p^m)w_\beta uh, s\Big] = \widetilde{f}_\chi\Big[h(1,p^m)w_\beta h, s\Big]$$

for all $h \in \widetilde{G}_2$. We also used the relation $w_\beta x_{\alpha+\beta}(pr_2)w_\beta = x_\alpha(pr_2)$. Conjugating $h(p^{-1},p^2)$ to the left, we get

$$\begin{aligned} R_1(m) &= \int_F \widetilde{f}_\chi\left[h(p^2,1)h(1,p^{m-1})w_\beta x_\beta(p^3r_2), s\right]\psi\left(p^{m+1}r_2\right)(p^m,p)dr_2 \\ &= q^{-12s+3}\chi(p^2)\int_F \widetilde{f}_\chi\Big[h(1,p^{m-1})w_\beta x_\beta(r_2), s\Big]\psi\left(p^{m-2}r_2\right)(p^m,p)dr_2. \end{aligned}$$

In the last equality we changed variables in r_2 and used the fact that $h(p^2,1)$ is in the center of the Levi part of P (as in Lemmas 3.2 and 3.5). Hence

$$\begin{aligned} R_1(m) =&\chi^2(p)q^{-12s+3}(p^m,p)\Big[\widetilde{f}_\chi(h(1,p^{m-1}))\int_{|r_2|\le 1}\psi(p^{m-2}r_2)dr_2 \\ &+\int_{|r_2|>1}\widetilde{f}_\chi\Big[h(1,p^{m-1})h(r_2^{-1},r_2), s\Big]\psi(p^{m-2}r_2)dr_2\Big] \\ =&\chi^2(p)q^{-3sm-9s+3}(p^m,p)\Big[W_\chi\begin{pmatrix} p^{m-1} & \\ & 1\end{pmatrix}\gamma_{p^{m-1}}^{-1}\int_{|r_2|\le 1}\psi(p^{m-2}r_2)dr_2 \\ &+\int_{|r_2|>1}W_\chi\begin{pmatrix} p^{m-1} & \\ & r_2^{-1}\end{pmatrix}|r_2|^{-3s}\gamma_{p^{m-1}r_2^{-1}}^{-1}\psi\left(p^{m-2}r_2\right)dr_2\Big] \\ =&\chi^2(p)q^{-3sm-9s+3}(p^m,p)\gamma_{p^{m-1}}^{-1}\Big[W_\chi\begin{pmatrix} p^{m-1} & \\ & 1\end{pmatrix}\int_{|r_2|\le 1}\psi\left(p^{m-2}r_2\right)dr_2 \\ &+\int_{|r_2|>1}W_\chi\begin{pmatrix} p^{m-1}r_2 & \\ & 1\end{pmatrix}\chi(r_2)^{-1}\left(p^{m-1},r_2\right)|r_2|^{-3s}\psi(p^{m-2}r_2)dr_2\Big] \\ =&\chi^2(p)q^{-3sm-9s+3}(p^m,p)\gamma_{p^{m-1}}^{-1}\Big[W_\chi\begin{pmatrix} p^{m-1} & \\ & 1\end{pmatrix}\int_{|r_2|\le 1}\psi\left(p^{m-2}r_2\right)dr_2 \\ &+\sum_{k=1}^{\infty}\chi(p)^k(p^{m-1},p^k)q^{(-3s+1)k}W_\chi\begin{pmatrix} p^{m-k-1} & \\ & 1\end{pmatrix} \\ &\times\int_{|\varepsilon|=1}(p^k,\varepsilon)\psi(p^{m-k-2}\varepsilon)d\varepsilon.\Big] \end{aligned} \tag{3.7}$$

Now we have:

LEMMA 3.7:

$$R_1(2n) = \begin{cases} 0, & n = 0, \\ q^{(-12s+2)n-6s+3/2}\chi(p)^{2n+1}, & n \geq 1. \end{cases}$$

Proof: Plug $m = 2n$ in (3.7) to obtain

$$R_1(2n) = \chi^2(p)q^{-6sn-9s+3}\gamma_p^{-1}\Big[W_\chi\begin{pmatrix} p^{2n-1} & \\ & 1\end{pmatrix}\int\limits_{|r_2|\leq 1}\psi(p^{2n-2}r_2)dr_2$$
$$+\sum_{k=1}^{\infty}\chi(p)^k(p,p^k)q^{(-3s+1)k}W_\chi\begin{pmatrix} p^{2n-k+1} & \\ & 1\end{pmatrix}$$
$$\int\limits_{|\varepsilon|=1}(p^k,\varepsilon)\psi\left(p^{2n-k-2}\varepsilon\right)d\varepsilon\Big].$$

The first term in the brackets vanishes for all n, since $2n-1$ is odd. The second term vanishes for $n = 0$. Thus $R_1(0) = 0$. Also, $\int_{|\varepsilon|=1} = 0$ if $k \equiv 0(2)$. Thus for $k \equiv 1(2)$,

$$\int\limits_{|\varepsilon|=1}(p^k,\varepsilon)\psi\left(p^{2n-k-2}\varepsilon\right)d\varepsilon = \int\limits_{|\varepsilon|=1}(p,\varepsilon)\psi\left(p^{2n-k-2}\right)d\varepsilon$$

which is zero unless $2n-k-2 = 1$ or $k = 2n-1$. Hence, for $n \geq 1$,

$$R_1(2n) = \chi^2(p)q^{-6sn-9s+3}\gamma_p^{-1}\chi(p)^{2n-1}q^{(-3s+1)(2n-1)}(p,p)q^{-1}G(p).$$

It follows from [10] that $\gamma_p G(p) = q^{1/2}$, and hence for $n \geq 1$ that

$$R_1(2n) = q^{(-12s+2)n-6s+3/2}\chi(p)^{2n+1}. \qquad \blacksquare$$

LEMMA 3.8:

$$R_1(2n+1) =$$
$$\begin{cases} 0, & n=0, \\ \frac{1-\chi(p)q^{-6s+3/2}}{1-\chi(p)q^{-6s+5/2}}(p,p)\chi(p)^{n+2}q^{-(6s+1/2)n-12s+3}(1-\chi^n(p)q^{(-6s+5/2)n}), & n \geq 1. \end{cases}$$

Proof: In (3.7) let $m = 2n+1$. We get

$$R_1(2n+1) = \chi^2(p)q^{-6sn-12s+3}(p,p)\Big[W_\chi\begin{pmatrix} p^{2n} & \\ & 1\end{pmatrix}\int\limits_{|r_2|\leq 1}\psi(p^{2n-1}r_2)dr_2$$
$$+\sum_{k=1}^{\infty}\chi(p)^k q^{(-3s+1)k}W_\chi\begin{pmatrix} p^{2n-k} & \\ & 1\end{pmatrix}\int\limits_{|\varepsilon|=1}(p^k,\varepsilon)\psi\left(p^{2n-k-1}\varepsilon\right)d\varepsilon\Big].$$

Each term vanishes for $n = 0$, hence $R_1(1) = 0$. Also

$$W_\chi \begin{pmatrix} p^{2n-k} & \\ & 1 \end{pmatrix} = 0 \quad \text{for } 2n < k \text{ or } k \equiv 1(2).$$

Thus, using (3.4), we get for $n > 1$

$$\begin{aligned} &R_1(2n+1) \\ &= \chi^2(p) q^{12s-6sn+3}(p,p)\Big[q^{-\frac{n}{2}}\chi(p)^n + \sum_{k=1}^{n} \chi(p)^{2k} q^{(-6s+2)k-\frac{n-k}{2}} \chi(p)^{n-k} \\ &\qquad \int_{|\varepsilon|=1} \psi\left(p^{2n-2k-1}\varepsilon\right) d\varepsilon\Big] \\ &= q^{-12s-(6s+1/2)n+3}\chi(p)^{n+2}(p,p)\Big[1 + (1-q^{-1})\sum_{k=1}^{n-1} \chi(p)^k \cdot q^{(-6s+5/2)k} \\ &\qquad - \chi(p)^n q^{(-6s+5/2)n-1}\Big], \end{aligned}$$

where in the last equality we need the identity

$$\int_{|\varepsilon|=1} \psi(p^i\varepsilon) d\varepsilon = \begin{cases} 1-q^{-1}, & i \geq 0, \\ -q^{-1}, & i = -1 \, . \end{cases}$$

Using the formula for geometric sums, the lemma follows. ∎

We proceed with $R_2(m)$. We have

$$R_2(m) = \int_F \int_{|r_1|>1} \widetilde{f}_\chi\Big[h(1,p^m) w_\beta w_\alpha x_\alpha(p^{-1}) x_{3\alpha+\beta}(r_2) x_\beta(r_1^{-1}) h(r_1^{-1}, r_1), s\Big] dr_1 dr_2.$$

Using the relation

$$x_\alpha(p^{-1}) x_\beta(r_1^{-1}) = x_\beta(r_1^{-1}) x_\alpha(p^{-1}) x_{2\alpha+\beta}\left(p^{-2} r_1^{-1}\right) u,$$

where u is a unipotent matrix in P, we obtain in a similar way as in the computation of $R_1(m)$

$$\begin{aligned} R_2(m) &= \int_F \int_{|r_1|>1} \widetilde{f}_\chi\Big[h(1,p^m) w_\beta w_\alpha x_\alpha(p^{-1}) x_{3\alpha+\beta}(r_2) h(r_1^{-1}, r_1), s\Big] \\ &\qquad \psi\left(p^{m-2} r_1^{-1}\right) dr_1 dr_2 \\ &= \int_F \int_{|r_1|>1} \widetilde{f}_\chi\Big[h(1,p^m) h(r_1^{-1},1) w_\beta x_\beta(r_2 r_1^{-1}) w_\alpha x_\alpha(p^{-1} r_1^{-1}), s\Big] \\ &\qquad \psi(p^{m-2} r_1^{-1}) dr_1 dr_2. \end{aligned}$$

Since $|r_1| > 1$, $x_\alpha(p^{-1}r_1^{-1})$ is in the maximal compact of G. Using this and the fact that $h(r_1^{-1}, 1)$ is in the center of the Levi part of P, we obtain

$$R_2(m) = \int_F \widetilde{f}_\chi\Big[h(1,p^m)w_\beta x_\beta(r_2), s\Big]dr_2$$

$$\times \int_{|r_1|>1} |r_1|^{-6s+1}\chi(r_1)^{-1}\gamma_{r_1^{-1}}\psi\left(p^{m-2}r_1^{-1}\right)dr_1$$

$$=I(m)\sum_{k=1}^{\infty} q^{(-6s+2)k}\chi(p)^k\gamma_{p^k}\int_{|\varepsilon|=1}(p^h,\varepsilon)\psi(p^{m+\ell-2}\varepsilon)d\varepsilon\ . \tag{3.8}$$

The last lemma is:

LEMMA 3.9:

$$R_2(2n+1) = 0;$$

$$R_2(2n) = \begin{cases} I(0)[q^{-6s+3/2}\chi(p) + \frac{(1-q^{-1})\chi^2(p)q^{-12s+4}}{1-\chi^2(p)q^{-12s+4}}], & n=0,\\ I(2n)\frac{(1-q^{-1})\chi^2(p)q^{12s+4}}{1-\chi^2(p)q^{-12s+4}}, & n\geq 1.\end{cases}$$

Proof: From Lemma 3.5, $I(2n+1) = 0$. Hence by (3.8), $R_2(2n+1) = 0$. Put $m = 2n$ in (3.8):

$$R_2(2n) = I(2n)\sum_{k=1}^{\infty} q^{(-6s+2)k}\chi(p)^k\gamma_{p^k}\int_{|\varepsilon|=1}(p^k,\varepsilon)\psi(p^{2n+k-2}\varepsilon)d\varepsilon.$$

The integration on ε vanishes if $2n + k - 2 < -1$. If $n = 0$,

$$R_2(0) = I(0)\Big[q^{(-6s+2)}\chi(p)\gamma_p\int_{|\varepsilon|=1}(p,\varepsilon)\psi(p^{-1}\varepsilon)d\varepsilon$$

$$+\sum_{k=2}^{\infty} q^{(-6s+2)k}\chi(p)^k\gamma_{p^k}\int_{|\varepsilon|=1}(p^k,\varepsilon)d\varepsilon\Big].$$

The terms corresponding to $k \equiv 1(2)$ vanish. Using the relation $\gamma_p G(p) = q^{1/2}$ we get

$$R_2(0) = I(0)\Big[q^{-6s+2/2}\chi(p) + (1-q^{-1})\sum_{k=1}^{\infty} q^{(-12s+4)k}\chi(p)^{2k}\Big].$$

In a similar way we compute $R_2(2n)$ for $n \geq 1$. ∎

This completes the proof of Lemma 3.4 and the unramified computation.

3.2. CONVERGENCE AND NONVANISHING. We start with

LEMMA 3.10: *Let $W \in \mathcal{W}(\pi,\psi)$, $\widetilde{f}_\chi \in I(\mathcal{W}_{\theta_\chi}, s)$ and $\phi \in \mathcal{S}(F)$. Then the integral $I(W,\phi,\widetilde{f}_\chi,s)$ converges absolutely for* Re(s) *large.*

Proof: Writing the Iwasawa decomposition in SL_2 and using the $K(\widetilde{G})$ finiteness of $\widetilde{f}_\chi$ and the $K(SL_2)$ finiteness of W and ϕ, we see as in the first steps of the proof of Proposition 3.1 that it is enough to prove the absolute convergence of

$$\int_{F^*}\int_{F^4} W\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix}\phi(r_1+t)|t|^{-5/2}\gamma_t\widetilde{f}_\chi \tag{3.9}$$
$$\times\left[h(1,t)w_0x_\alpha(r_1)x_{2\alpha+\beta}(r_2)x_{3\alpha+\beta}(r_3)x_{3\alpha+2\beta}(r_4),s\right]dr_id^*t.$$

Since W is fixed by some small compact open subgroup of SL_2, we see that

$$W\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix}=0 \quad \text{if } |t| \text{ is large.}$$

Since $\phi\in\mathcal{S}(F)$, $\phi(r_1+t)=0$ if $|r_1+t|$ is large. Thus we may deduce that (3.9) is zero if $|r_1|$ is large. Using the $K(\widetilde{G})$ finiteness of $\widetilde{f}_\chi$ it is enough to study the absolute convergence of

$$\int_{F^*}\int_{F^3} W\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix}|t|^{-5/2}\gamma_t\widetilde{f}_\chi$$
$$\times\left[h(1,t)w_0x_{2\alpha+\beta}(r_2)x_{3\alpha+\beta}(r_3)x_{3\alpha+2\beta}(r_4),s\right]dr_id^*t.$$

Conjugating w_0 to the right implies that we may study the absolute convergence of

$$\int_{F^*}\int_{F^3} W\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix}|t|^{-5/2}\gamma_t\widetilde{f}_\chi \tag{3.10}$$
$$\times\left[h(1,t)x_{-(\alpha+\beta)}(r_2)x_{-(3\alpha+2\beta)}(r_3)x_{-\beta}(r_4),s\right]dr_id^*t.$$

Next we write the Iwasawa decomposition of $x_{-(\alpha+\beta)}(r_2)x_{-(3\alpha+2\beta)}(r_3)x_{-\beta}(r_4)$. We do this by breaking the domain of integration into eight separate cases either $|r_i|<c_i$ or $|r_i|\ge c_i$ for some constants c_i with $i=2,3,4$. By $K(\widetilde{G})$-finiteness of $\widetilde{f}_\chi$ we may ignore the integration on those variables r_i with $|r_i|<c_i$. Let us treat the case where all $|r_i|\ge c_i$. In this case, the contribution to (3.10), provided c_i

is large enough, is

$$\int_{F^*}\int_{|r_i|>c_i} W\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix}|t|^{3s-5/2}\gamma_{tr_2^{-3}r_3^{-2}r_4^{-1}} \tag{3.11}$$
$$\times W_\chi\begin{pmatrix} r_2^{-1}r_3^{-1}t & \\ & r_4^{-1}r_2^{-2}r_3^{-1}\end{pmatrix}|r_2^9r_3^6r_4^3|^{-s}dr_id^*t.$$

The absolute convergence of this integral for Re(s) large follows from the usual estimation of Whittaker functions (see [6]). ∎

Since $I(W,\phi,\widetilde{f}_\chi,s)$ is a finite sum of integrals of the type of (3.11), it also follows from the asymptotic expansion of the Whittaker functions that $I(W,\phi\widetilde{f}_\chi,s)$ is a rational function in q^{-s}. Thus we have:

LEMMA 3.3: *$I(W,\phi,\widetilde{f}_\chi,s)$ is a rational function in q^{-s}. In particular, it admits a meromorphic continuation to the whole complex plane.*

Finally, we prove

PROPOSITION 3.4: *There exists a choice of data such that given $s_0\in\mathbb{C}$, the integral $I(W,\phi,\widetilde{f}_\chi,s)$ is nonzero at $s=s_0$.*

Proof: We argue in a similar way as in [13] section 6. Choose $W\in\mathcal{W}(\pi,\psi)$ and $W_\chi\in\mathcal{W}(\theta_\chi,\psi)$ such that $W(e)W_\chi(e)\neq 0$. For Re(s) large, we have

$$I\left(W,\phi,w_0^{-1}\widetilde{f}_\chi,s\right)=$$
$$\int_{F^*}\int_{F^5} W\left(\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ r_5 & 1\end{pmatrix}\right)\omega_\psi\left[\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix}\begin{pmatrix}1 & \\ r_5 & 1\end{pmatrix}\right]\phi(r_1+1)$$
$$\widetilde{f}_\chi\left(w_0x_\alpha(r_1)x_{2\alpha+\beta}(r_2)x_{3\alpha+\beta}(r_3)x_{3\alpha+2\beta}(r_4)h(t,t^{-1})x_{-\beta}(r_5)w_0^{-1},s\right)$$
$$\psi(r_2)|t|^{-2}dr_id^*t\ .$$

Conjugating $h(t,t^{-1})$ to the left in $\widetilde{f}_\chi$, we obtain

(3.12)

$$I\left(W,\phi,w_0^{-1}f_\chi,s\right)=\int_{F^*}\int_{F^5} W\left(\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ r_5 & 1\end{pmatrix}\right)\omega_\psi\begin{pmatrix}1 & 0\\ r_5 & 1\end{pmatrix}\phi(r_1+t)\widetilde{f}_\chi$$
$$\left(h(1,t)w_0x_\alpha(r_1)x_{2\alpha+\beta}(r_2)x_{3\alpha+\beta}(r_3)x_{3\alpha+2\beta}(r_4)x_{-\beta}(r_5)w_0^{-1},s\right)$$
$$\psi(r_2)(t,t)\gamma_t|t|^{-5/2}dr_id^*t\ .$$

To obtain the above, we also used a suitable change of variables. Given $W \in \mathcal{W}(\pi,\psi)$ and $\phi \in \mathcal{S}(F)$ there is a compact open subgroup K^0 of SL_2 such that $W(gk^0) = W(g)$ for $g \in \mathrm{SL}_2$ and $\omega_\psi(k^0)\phi = \phi$ for all $k^0 \in K^0$. Also for $|r_1|$ small enough, $\omega_\psi((r_1,0,0))\phi = \phi$. Let K^1 be a sufficiently small compact open subgroup of $\widetilde{G}$ such that $x_{\alpha+\beta}(r_1)$, $x_\beta(r_5) \in K^1$ whenever $\begin{pmatrix} 1 & \\ r_5 & 1 \end{pmatrix} \in K^0$ and $\omega_\psi((r_1,0,0))\phi = \phi$. Thus K^1 depends on the choice of W and ϕ. Let $u_1 = x_\alpha(r_1)x_{2\alpha+\beta}(r_2)x_{3\alpha+\beta}(r_3)x_{3\alpha+2\beta}(r_4)x_{-\beta}(r_5)$. We claim that if $w_0u_1w_0^{-1} \in PK^1$ then $w_0u_1w_0^{-1} \in K^1$. To see this we use matrix multiplication. Realize G_2 as a subgroup of SO_7. One can choose the embedding so that P will be contained in the maximal parabolic subgroup of SO_7 whose Levi part is $\mathrm{GL}_2 \times \mathrm{SO}_3$. In this embedding one has

$$w_0u_1w_0^{-1} = \begin{pmatrix} I_2 & & \\ X & I_3 & \\ Z & X^* & I_2 \end{pmatrix}$$

where X, Z and X^* are so that the above matrix is in G_2. Also for $p \in P$ we have

$$p = \begin{pmatrix} A & * & * \\ & B & * \\ & & A^* \end{pmatrix} \quad \text{where } A \in \mathrm{GL}_2 \text{ and } B \in \mathrm{SO}_3,$$

in such a way that p is in G_2. Thus a simple matrix multiplication of $(w_0uw_0^{-1})p$ shows that $w_0u_1w_0^{-1}p \in K^1$ implies that $w_0u_1w_0^{-1} \in K^1$. Choose $\widetilde{f}_\chi$ which is supported on PK^1. Write also

$$\omega_\psi\begin{pmatrix} 1 & \\ r_5 & 1 \end{pmatrix}\phi(t+r_1) = \omega_\psi\left[(r_1,0,0)\begin{pmatrix} 1 & \\ r_5 & 1 \end{pmatrix}\right]\phi(t).$$

Thus (3.12) equals, up to a nonzero constant,

$$\int_{F^*} W\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}\phi(t)W_\chi\begin{pmatrix} t & \\ & 1 \end{pmatrix}|t|^{3s+a}(t,t)dt$$

where $a \in \mathbb{Z}$. Choose ϕ to be supported on the set $1 + \mathcal{P}^m$. Then if m is large enough, the above integral is a nonzero constant times $W(e)W_\chi(e)$. This completes the proof. ∎

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